U Subsitution

U Substitution for Integrals

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Outline

- Integration of Substitution
- Change of Variables
- Theorem 1.7: Substitution with Indefinite Integrals
  Let $u = g(x)$, where $g'(x)$ is continuous over an interval, let $f(x)$ be continuous over the corresponding range of g, and let $F(x)$ be an antiderivative of $f(x)$. 1.8 Then,
  $$\int f([g(x)]g'(x)\,dx = \int f(u)\,du = F(u) + C = F(g(x)) + C$$

Proof:

Let f, g, u, and F be as specified in the theorem. Then

$$\frac{dF(g(x))}{dx} = F'(g(x))g'(x) = f[g(x)]g'(x)$$

Integrating both sides with respect to x, we see that

$$\int f[g(x)]g'(x)\,dx = F(g(x)) + C$$

1.7 Theorem Examples

If we now substitute $u = g(x)$, and $du = g'(x)\,dx$, we get

$$\int f[g(x)]g'(x)\,dx = \int f(u)\,du = F(u) + C = F(g(x)) + C$$

Look at the original problem $f(x) = x^3$, $g(x) = x^2 - 3$, and $g'(x) = 2x$. Then,

$$f[g(x)]g'(x) = (x^2 - 3)^3(2x)$$

we let $u = x^2 - 3$, and then $du = 2x\,dx$, rewrite in term of u

$$\int \underbrace{(x^2 - 3)^3}_{\text{u}}(2 \underbrace{xd}_{\text{du}} x) = \int u^3\,du$$

Using the power rule for integrals, we have

$$\int (u^3) du = \frac{u^4}{4} + C$$

Substitute the original expression for x back into the solution:

$$\frac{u^4}{4} + C = \frac{(x^2-3)^4}{4} + C$$

Click the Link

1.7 Therem Examples

Example 1.30
Use subsitution to find the antiderivative of $\int z\sqrt{z^2-5}\,dz$ Solution:
Let $u = z^2 - 5$, then $du = 2zdz$, Thus,

$$u = z^2 - 5 \;\; du = 2zdz \;\; \frac{1}{2}du = \frac{1}{2}(2z)dz = zdz$$

Then

$$\int z(z^2-5)^{1/2}dz = \frac{1}{2}\int u^{1/2}du$$

Integrate the expression in u:

$$\frac{1}{2}\int u^{1/2}du = \left(\frac{1}{2}\right)\frac{u^{3/2}}{3/2} + C = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)u^{3/2} + C = \frac{1}{3}u^{3/2} + C$$
$$= \frac{1}{3}(z^2-5)^{3/2} + C$$

Example 1.31 Using substitution to evaluate the integral $\int \frac{sint}{cos^3 t}\, dt$.
Solution:
let $u = cost$, then

$$\int \frac{sint}{cost^3 t} dt = -\int \frac{du}{u^3}$$

Evaluating the integral,

$$-\int \frac{du}{u^3} = -\int u^{-3} du = -(-\frac{1}{2})u^{-2} + C$$

Putting the ansiwer back, we get

$$\int \frac{sint}{cost^3 t} dt = \frac{1}{2u^2} + C = \frac{1}{2cos^2 t} + C$$

- Theorem 1.8: Substitution with Definite Integrals
  let $u = g(x)$ and let g be continuous over an interval [a, b], and let f be continuous over the range of $u = g(x)$, Then, ▸ 1.8
  $$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

1.8 Theorem Examples

When

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C$$

Then

$$\int_a^b f(g(x))g'(x)\,dx = F(g(x))\Big|_{x=a}^{x=b} = F(g(b)) - F(g(a)) = F(u)\Big|_{u=g(a)}^{u=g(b)} = \int_{g(a)}^{g(b)} f(u)\,du$$

Example 1.34 Use substitution to evaluate $\int_0^1 x^2(1+2x^3)^5\,dx$
Solution:
$u = 1 + 2x^3$, so $du = 6x^2\,dx$, Then

$$\frac{1}{6} = x^2 dx$$

then

$$\int_0^1 x^2(1+2x^3)^5\,dx = \frac{1}{6}\int_1^3 u^5 du$$

Evaluate it,

$$\frac{1}{6}\int_1^3 u^5 du = (\frac{1}{6})(\frac{u^6}{6})|_1^3 = \frac{1}{36}[3^6 - 1^6] = \frac{182}{9}$$

1.8 Theorem Examples

Example 1.35 Use substitution to evaluate $\int_0^1 xe^{4x^2+3}\,dx$

Solution

Let $u = 4x^2 + 3$. Then, $du = 8xdx$. Use substitution, we have

$$\int_0 1xe^{4x^2+3}\,dx = \frac{1}{8}\int_3 7e^u du = \frac{1}{8}e^u|_3^7 = \frac{e^7-e^3}{8} = 134.568$$

Example 1.36 Use substitution to evaluate $\int_0^{\pi/2} cos\theta \, d\theta$

Solution

Use substitution, we have

$$\int_0^{\pi/2} cos\theta \, d\theta = \int_0^{\pi/2} \frac{1+\cos 2\theta}{2} d\theta$$

Then,

$$\int_0^{\pi/2} cos^2\theta \, d\theta = \int_0^{\pi/2} (\tfrac{1}{2} + \tfrac{1}{2}\cos(2\theta)) d\theta = \tfrac{1}{2}\int_0^{\pi/2} \, d\theta + \tfrac{1}{2}(\tfrac{1}{2})\int_0^{\pi} cosudu = \tfrac{\theta}{2}\Big|_{\theta=0}^{\theta=\pi/2} + \tfrac{1}{4}\sin u\Big|_{u=0}^{u=\theta} = (\tfrac{\pi}{4} - 0) + (0 - 0) = \tfrac{\pi}{4}$$

1.8

theorem

1.7 Substitution for Indefinite Integral ▸ 1.7.1
1.8 Subsitution for Definite Integral ▸ 1.7.2

Exercise 1:

Iff $f = g \circ h$, when reversing the chain rule, we have $\frac{d}{dx}(g \circ h)(x) = g'(h(x))h'(x)$, which function is u, $g(x)$, or $h(x)$?

Exercise 2:

for $\int x\sqrt{x+1}\,dx$, can we suppose $u = x + 1$? Why or why not?

Exercise 3:

Use substituion to find the antiderivative of $\int x^2(x^3+5)^9\,dx$

Exercise 4:

Use substitution to evaluate $\int_0^1 x^2 \cos(\frac{\pi}{2}x^3)\,dx$

Summary

- The U substitution of indefinite integrals.
- The U substitution of definite integrals.
- From Exercise 2 and 3, do you find any pattern for finding u to replace to use U-substitution?.
- Moreover, for integral that contains trignomitric functions, can you find a pattern for carrying out U-substitution?
- Outlook
  - What things haven't you solved?
  - What else haven't you solved ?

For Further Reading I

Calculus II.
OpenStax 2017.