2. The Definite Integral

(Notes)

2.1 Introduction

In this section, we will further discuss integral. This time, function $f(x)$ isn't necessary a possitive function. Is the limit method still avaible to find integral of the function? What about the geometric meaning of integral when function is posstive and what is the geometric meaning when it is negative? As a funciton and its interval are given, how can we find its integral? Those questions will be discussed in this section. In the future time, we will discuss how to find arc length, volume of s solid, force, etc.

2.2 Definition of Definte Integral

Riemann sum

1. Definition 2:
   If f is a function defined for x in [a,b], we divide [a,b] into n subintervals of equal width $\triangledown x = \frac{b-a}{n}$. We let $x_0, x_1, x_2, ...x_n$ be the endpoints of these subinterals and we let x_1^* be any sample points in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the definite integral of f from a to b is

   $$\int_a^b f(x)\,dx = A_1 - A_2$$

   Provided the limit exists, we say that f is integrable on [a,b]
   Remark:
   1). About Riemann
   2). Riemann sum unique features: Riemann sum contains upper limit and lower limit; Riemann sum is a number, not dependable on x; Rieman sum is a net area.

2. Net Area: In the notation $\int_a^b f(x)\,dx$, integrand is $f(x)$. Riemman sum is a net area, not area. We know about the point from three cases in the following.
   1). If $f(x)$ happens to be positive, the definition of Riemman sum has no difference from previous definition of area. Thus, to find definite integral or Riemman sum is to find area under the curve $f(x)$ from a to b (figure2).
   2). If $f(x)$ takes on positive and negative values, the area A_1 of region that is above the axis is positive, and the area A_2 of the region that is below the axis is negative, and the Riemman sum called net area is $A_1 - A_2$, and we can denote as

   $$\int_a^b f(x)\,dx = A_1 - A_2$$

   3). If subintervals don't have equal width, we will have a formula such as

   $$\int_a^b f(x)\,dx = \lim_{max \triangle x_i \to +\infty} \sum_{i=1}^{n} f(x_i^*) \triangle x_i$$

3. Theorem3: If f is continuous on [a,b], or if f has only a finite number of jump discountinuities, then f is integrable on [a,b]; that is, the definite integral $\int_a^b f(x)\,dx$ exists

4. Theorem4: If f is integrable on [a,b], then

   $$\int_a^b f(x)\,dx = \lim_{n \to +\infty} \sum_{i=1}^{n} f(x_i) \triangle x$$

   where

   $$\triangle x = \frac{b-a}{n} \text{ and } x_i = a + i \triangle x$$

5. **Example 1:**
Writing a limit of Riemann sums as an integral Express

$$\lim_{n\to+\infty}\sum_{i=1}^{n}(x_i^3+x_i)\sin x_i \bigtriangleup x$$

as an integral on the interval $[0,\pi]$

Solution:
In this question, we know $f(x_i)=x_i^3+\sin x_i$, by theorem 4, $f(x)=f(x_i)$, thus, $f(x)=x^3+\sin x$ and $dx=\triangle x$. Given $a=0, b=\pi$, we have $\lim_{n\to+\infty}\sum_{i=1}^{n}(x_i^3+x_i)\sin x_i \bigtriangleup x=\int_0^{\pi} f(x)\,dx$

6. Exercise1: textbook p354 17

7. Remark:
In general, we replace x_i by x_i^* , we will write the above equation as

$$\int_a^b f(x)\,dx=\lim_{n\to+\infty}\sum_{i=1}^{n} f(x_i^*)\bigtriangleup x$$

2.3 Evaluating Integrals

1. Useful Formulas

Rule: Sums and Powers of Integers

1. The sum of n integers is given by

$$\sum_{i=1}^{n} i = 1+2+\cdots+n=\frac{n(n+1)}{2}.$$

2. The sum of consecutive integers squared is given by

$$\sum_{i=1}^{n} i^2 = 1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}.$$

3. The sum of consecutive integers cubed is given by

$$\sum_{i=1}^{n} i^3 = 1^3+2^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}.$$

Figure 1:

2. **Example 2: Evaluating an integral as a limit of Riemann sums**

(a) Evaluate the Riemann sum for $f(x)=(x^3-6x)$, taking the sample points to be right endpoints and $a=0, b=3$, and $n=6$

(b) Evaluate $\int_0^3 x^3-6x\,dx$

(c) solution:

i. When $n=6$, the interval width is

$$\triangle x=\frac{b-a}{n}=\frac{3-0}{b}=\frac{1}{2}$$

the right endpoints are $x_1=0.5, x_2=1.0, x_3=1.5, x_4=2.0, x_5=2.5$, and $x_6=3.0$. So the Riemann sum is

$$\begin{aligned} A &= \lim_{n\to+\infty} R_n \\ &= \lim_{n\to+\infty}[f(x_1)\bigtriangleup x+f(x_2)\bigtriangleup x+...+f(x_n)\bigtriangleup x] \\ &= \frac{1}{2}(-2.875-5-5.625-4+0.625+9) \\ &= -3.9375 \end{aligned}$$

ii. With n subintevals we have

$$\triangle x=\frac{b-a}{n}=\frac{3-0}{n}=\frac{3}{n}$$

Thus, $x_0 = 0, x_1 = \frac{3}{n}, x_2 = \frac{6}{n}$, and in general, $x_i = \frac{3i}{n}$. We use right endpoints, by Theorem 4, we have the follows:

$$\begin{aligned}
\int_a^b f(x)\,dx &= \lim_{n\to+\infty} \sum_{i=1}^{n} f(x_i) \bigtriangleup x_i \\
&= \lim_{n\to+\infty} \sum_{i=1}^{n} f(\frac{3i}{n})\frac{3}{n} \\
&= \lim_{n\to+\infty} \sum_{i=1}^{n} [(\frac{3i}{n})^3 - 6(\frac{3i}{n})]\frac{3}{n} \\
&= \lim_{n\to+\infty} \sum_{i=1}^{n} [(\frac{27}{n^3} \times i^3) - \frac{18}{n} \times i]\frac{3}{n} \\
&= \lim_{n\to+\infty} [\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i] \\
&= \lim_{n\to+\infty} \{\frac{81}{n^4}[\frac{n(n+1)}{2}]^2 - \frac{54}{n^2}\frac{n(n+1)}{2}\} \\
&= \lim_{n\to+\infty} [\frac{81}{4}(1+\frac{1}{n})^2 - 27(1+\frac{1}{n})] \\
&= \frac{81}{4} - 27 \\
&= -\frac{27}{4} \\
&= -6.75
\end{aligned}$$

(d) Remark:
i) When increasing the number of n, for example, $n = 40$, we find the table below shows the better estimating for Riemanne sum.

n	40	100	500	1000	5000
R_n	-6.3998	-6.6130	6.7229	-6.7365	-6.7471

ii) From the graph, this integral cannot be interpreted as an area of the function f because f takes on both positive and negative values. Therefore, this integral is the net area of the function f, and denoted as $A_1 - A_2$

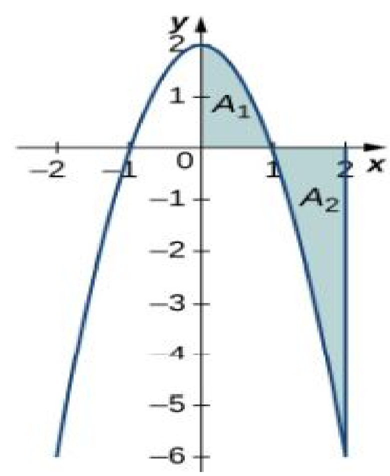

Figure 2:

3. **Example 3:**

(a) Set up an expression for $\int_1^3 e^3\,dx$ as a limit of sums
(b) Use a computer aslgebra system to evaluate the expression
Solution:

i. $f(x) = e^x, a = 1, b = 3$, and
$\triangle = \frac{b-a}{n} = \frac{2}{n}$,
Thus, $x_0 = 1, x_1 = 1 + \frac{2}{n}, x_2 = 1 + \frac{4}{n}, x_3 = 1 + \frac{6}{n}$, and

$$x_i = 1 + \frac{2i}{n}$$

from Theorem 4, we have

$$\begin{aligned}\int_1^3 e^x\,dx &= \lim_{n\to+\infty} \sum_{i=1}^{n} f(x_i) \triangle x \\ &= \lim_{n\to+\infty} \sum_{i=1}^{n} f(1 + \frac{2i}{n})\frac{2}{n} \\ &= \lim_{n\to+\infty} \frac{2}{n} \sum_{i=1}^{n} e^{1+\frac{2i}{n}}\end{aligned}$$

ii. By computer algebra system, after simplifying, we obtain

$$\sum_{i=1}^{n} e^{1+\frac{2i}{n}} = \frac{e^{\frac{3n+2}{n}} - e^{\frac{n+2}{n}}}{e^{\frac{2}{n}} - 1}$$

Now, we find the solution by using computer algebra system

$$\int_1^3 e^x\,dx = \lim_{n\to+\infty} \frac{2}{n} = \frac{e^{\frac{3n+2}{n}} - e^{\frac{n+2}{n}}}{e^{\frac{2}{n}} - 1}$$

2.4 Geometric Evaluation Method

Example 4: Evaluate the following integrals by interpreting each in terms of areas:

1) $\int_3^6 \sqrt{(9 - (x-3)^2)}\,dx$

2) $\int_0^3 (x-1)\,dx$

Solution:

1) We find $f(x) = \sqrt{9 - (x-3)^2}$ is a positive function. Thus, this integral can be explained as an area under the curve $y = \sqrt{9 - (x-3)^2}$ from 3 to 6. Thus, we have $y^2 = 9 - (x-3)^2$, and $(x-3)^2 + y^2 = 9$, which shows the graph is a quater of a circle with center(3,0) and radius = 3

$$\int_3^6 \sqrt{9 - (x-3)^2}\,dx = \frac{4}{9}\pi \approx 7.069$$

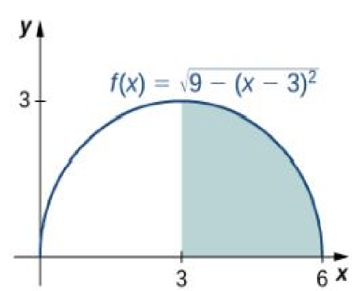

Figure 3:

2) The graph of $y = x - 2$ is the line with slope 1 shown in Figure. That is, the curve has part of graph above x-axis, and part of it below x-axis. We compute the integral as the difference of the areas of the two triangles. $\int_0^6 (x-2)\,dx = A_1 - A_2 = |\frac{1}{2}(2*2) - \frac{1}{2}(4*4)| = 10$

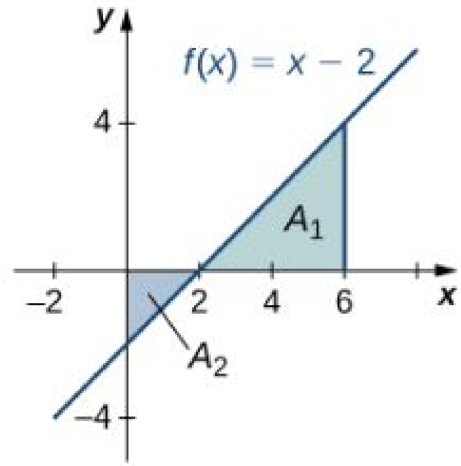

Figure 4:

2.4 The Midpoint Rule

To find an approaximation to an integral, we usually better choose x_i^* to be the midpoint of a subinterval, we denoted the point as $\bar{x}$

The Midpoint Rule
$\int_a^b f(x)\,dx = f(\bar{x}_1) \triangle x + f(\bar{x}_2) \triangle x + ... + f(\bar{x}_n) \triangle x$
where $\triangle x = \frac{b-a}{n}$
and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$= midpoint of $[x_{i-1}, x_i]$

Example 5
Use the Midpoint Rule with n = 5 to approximate $\int_1^2 \frac{1}{x}$
Solution:
Since a = 0, b = 1, the width of each subinteral is $\frac{1}{5}$. So the five subintervals are [1,1.2],[1.2,1.4],[1.4,1.6],[1.6,1.8],[1.8,2], their midpoints, repectively, are 1.1,1.3,1.5,1.7, and 1.9. Thus, the Midpoint Rule find its integral

$$\begin{aligned}\int_1^2 \frac{1}{x} &= f(1.1) \triangle x + f(1.3) \triangle x + ... + f(1.9) \triangle x \\ &= \frac{1}{5}(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}) \\ &= 0.691908\end{aligned}$$

2.5 Properties of the Defintie Integral

- Properties 1 - 5

  $\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$

  if $a = b$, then $\triangle x = 0$, so

  $\boxed{\int_a^a f(x)\,dx = 0}$

  Other properties:
  1) $\int_a^b c\,dx = c(b-a)$
  2) $\int_a^b (f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
  3) $\int_a^b (cf(x))\,dx = c\int_a^b f(x)\,dx$
  4) $\int_a^b (f(x) - g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

  Example 6 Use the properties of integral to evaluate $\int_0^1 (4 + 3x^2)\,dx$
  Solution: Use Properties 2 and 3 of integrals, we have

$$\begin{aligned}\int_0^1 (4+3x^2)\,dx &= \int_0^1 4\,dx + \int_0^1 3x^2\,dx \\ &= \int_0^1 4\,dx + 3\int_0^1 x^2\,dx\end{aligned}$$

from property 1, we have

$$\int_0^1 4\,dx = 4(1-0) = 4$$

from previous section Eample 2

$$\int_0^1 x^2\,dx = \frac{1}{3}$$

So

$$\begin{aligned}\int_0^1 (4+3x^2)\,dx &= \int_0^1 4\,dx + \int_0^1 3x^2\,dx \\ &= \int_0^1 4\,dx + 3\int_0^1 x^2\,dx \\ &= 4 + 3\cdot\frac{1}{3} & = 5\end{aligned}$$

- Properties 6

  $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b (f(x) + f(x)\,dx$

  Example 7
  If it is known that $\int_0^{10} f(x)\,dx = 17$, and $\int_0^8 f(x) = 12$, find $\int_8^{10} f(x)\,dx$.
  Solution: By Property 5,
  we have

  $$\int_0^8 f(x)\,dx + \int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx$$

- Properties 7 - 9
  7) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$
  8) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$
  9) If $m \geq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

  Example 8 Use Property 9 to estimate $\int_0^1 e^{-x^2}\,dx$
  Solution: In this question, we know $f(x) = e^{-x^2}$, this is a decreasing function from its graph ((insert a picture)) on [0,1]. From the graph, we can easily see the absolute maximum value is $M = f(0) = 1$ and absolute minimum value is $m = f(1) = e^{-1}$. Thus, by property 9,

  $$e^{-1}(1-0) \leq \int_0^1 e^{-x^2}\,dx \leq 1(1-0)$$

  Since $e^{-1} \simeq 0.3679$, thus, we can write

  $$0.367 \leq \int_0^1 e^{-x^2}\,dx \leq 1(1-0)$$

2.6 Conclusion

1) In this section, we have discussed using right-end method we previous learned to find limit of Riemann sum. The difference of the section from the last section about finding limit of sum in right-end point method lies on sign of function value, which we found the sum of the function areas is a net area. In other word, such area values sometimes are negative, not always positive.
2) Property 1 of definite integral tells us a change of definite integral in sign when we reverse the limits of the integral, but no change of the value of the integral.
3) Properties 2, 4 and 5 of definite integral provide us a method to simplify integral by breaking down an entire function $f(x)$ within its interval, and calculate ithe function's sub-integrals within each sub-interval
4) Properties 9 provides us a way to estimate a definite integral by identifing if a function's increasing or decreasing so that their absolute maximum(global maximum) or absolute minimum(global minimum) can be revealed.