1. Areas Between Curves

(Notes)

1.1 Introduction

Finding area between curves is a common problem we meet in the real world. There are lots of application problems that use integration to find area between two curves problems. In this section, we will calculate areas between simple curves in which one function is always above another function.

Consider the interval [a, b], We do partition, so p = x_i. Choosing a point x_i^* in $[x_{i-1}, x_i]$, the point corresponding vertically from $g(x_i^*)$ to $f(x_i^*)$.

To find the area problem, we have basic procedure in the following: 1. partition interval [a, b]
2. calcuate the length of each subinterval
3. select a method such as left endpoint method, find f and g values at point x_i^*
4. Integral $f(x^*) - g(x^*)$
where a rectangle length is $\triangle x$ and height $f(x^*) - g(x^*)$

Area of a Region

Example1: Use rectangles to estimate the area under the parabola $y = x^2$ from 0 and 1 as shown in the figure to the right.

1. Integral Estimation
   Idea: Suppose S is an area we want to estimate, we can divide S into pieces (lets say strips) depending on accuracy of approximation, which we need to find each strip's area using rectangle area formula. Thus, we need to know each strip's width and height. The area of each unit strip will be added to find its total area of the graph.

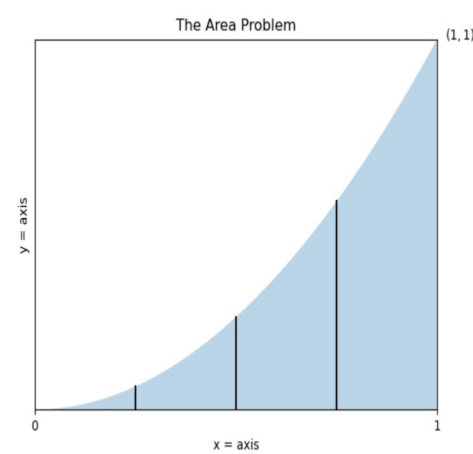

Figure 1:

   Solution: By the procedure of finding area
   Step 1: We divide the interval [0,1] into the subintervals $[0, \frac{1}{4}], [\frac{1}{4}, \frac{2}{4}], [\frac{2}{4}, \frac{3}{4}]$, and $[\frac{3}{4}, 1]$, when given n = 4
   Step 2: We calculate each height of small recatangel by plugging the rightmost endpoint of each subinterval into the formula $f(x) = x^2$ when selecting the right endpoint method; Thus, each height is $(\frac{1}{4})^2$, $(\frac{2}{4})^2$,$(\frac{3}{4})^2$ and 1^2 respectively.
   Step3: Suppose R_4 represent the area of approaximation, we have

$$R_4 = \frac{1}{4} \times (\frac{1}{4})^2 + \frac{1}{4} \times (\frac{2}{4})^2 + \frac{1}{4} \times (\frac{3}{4})^2 + \frac{1}{4} \times (1)^2 = \frac{15}{32} = 0.46875$$

Similarily, to use left endpoint method, we just plug the leftmost endpoint of each subinterval into the formula in the Step2. We denote the sum of rectangles areas as L_4, we have calculation as follows

$$L_4 = \frac{1}{4} \times (0)^2 + \frac{1}{4} \times (\frac{1}{4})^2 + \frac{1}{4} \times (\frac{2}{4})^2 + \frac{1}{4} \times (\frac{3}{4})^2 = \frac{7}{32} = 0.21875$$

We see the area with larger value R_4 is the upper area of the approximation, and the smaller value L_4 is the lower area of the approximation, which we form the follow expression

$$0.21875 \leq A \leq 0.46875$$

2. Left Endpoint Estimation
   Idea: We obtain only three strips in the graph. It is because the first one is collapsed when we find its height $f(0) = 0$ with other heights such as $f(\frac{1}{4}) = \frac{1}{16}$, and then $f(\frac{2}{4}) = \frac{4}{16}$,$f(\frac{3}{4}) = \frac{9}{16}$, we can calculate its area with the left endpoint approximation method
   Thus, let's use the method to approximate the area for example 1

   Solution: Suppose L_4 be area of left endpoint approximation. Since the first strip is collapsed when we find its height $f(0) = 0$, and other heights such as $f(\frac{1}{4}) = \frac{1}{16}$, then $f(\frac{2}{4}) = \frac{4}{16}$,$f(\frac{3}{4}) = \frac{9}{16}$, we have
   $L_4 = \frac{1}{4} \times 0 + \frac{1}{4} \times \frac{1}{16} + \frac{1}{4} \times \frac{4}{16} + \frac{1}{4} \times \frac{9}{16} = 0.21875$, which is smaller than the real one

3. Exercise
   If we choose n = 10, 20, ..., using either right endpoint approximation or left endpoint estimation to find area approximation. Establishing a table, we have the values available, now describe what conclusion can we draw from the table?

4. Discussion
   Let's review the process of finding area approximation using right endpoint and left endpoint method, please list three steps of using right endpoint method.

1.2 Integral Method

We have methods for approximating areas under a curve. Is there a method that accurately calculate area under the curve? After discussion, we approach a better and better estimation when increasing size of n for strips, just as described in the second chapter about limit concept. If we suppose there are n strips under the curve, what is the sum of the strips as n is a infinite large number.
Example 2: For the region S in Example 1, show that the sum of the areas of the upper approximating rectangles approaches $\frac{1}{3}$, that is,

$$\lim_{x \to +\infty} R_n$$

1. Right Endpoint Method Idea: As described above, the width of each strip is $\frac{1}{n}$, which therefore, divide the given interval [0, 1] into subinterval $[0, \frac{1}{n}], [\frac{1}{n}, \frac{2}{n}], ..., [\frac{n-1}{n}, 1$; to use Right Endpoint Method, we need to find height of each strip starting from the right end point of each interval, that is, $\frac{1}{n}, \frac{2}{n}, ... \frac{n}{n}$, we have $f(\frac{1}{n}) = (\frac{1}{n})^2 = \frac{1}{n^2}$. Thus, we have

$$\begin{aligned}
R_n &= \frac{1}{n} \times (\frac{1}{n})^2 + \frac{1}{n} \times (\frac{2}{n})^2 + \frac{1}{n} \times (\frac{3}{n})^2 + ... + \frac{1}{n} \times (\frac{n}{n})^2 \\
&= \frac{1}{n} \times (\frac{1}{n^2}) \times (1^2 + 2^2 + ... + n^2) \\
&= \frac{1}{n^2}(1^2 + 2^2 + ... + n^2) \\
&= \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\
&= \frac{(n+1)(2n+1)}{6n^2}
\end{aligned}$$

To find the best approximation, we have

$$\begin{aligned}
\lim_{x\to+\infty} R_n &= \lim_{x\to+\infty} \frac{(n+1)(2n+1)}{6n^2} \\
&= \lim_{x\to+\infty} \frac{1}{6}\frac{n+1}{n}\frac{2n+1}{n} \\
&= \lim_{x\to+\infty} \frac{1}{6}(1+\frac{1}{n})+(2+\frac{1}{n}) \\
&= \frac{1}{6}\cdot 1\cdot 2 \\
&= \frac{1}{3}
\end{aligned}$$

Note: This is how we find upper area approximation since we use Right Endpoint estimation.

2. Left Endpoint Method
   Please complete the Left Endpoint Estimation. What is the limit of the area by using the method? Is the area under approximate or over estimate?

3. Discussion
   Comparing the two methods, how do you define the area to be the limit of the sums of the areas of the approximating rectangles in math symbols?

4. Discussion
   If given an interval more generally, for example, the width of interval is [a, b] of the n strips, write a process of finding area using Right Endpoint method for the given subintervals $[x_0,x_1],[x_1,x_2],[x_3,x_4],......,[x_{n-1},x_n]$, where $x_0=a, x_n=b$.

1.3 Midpoint Method

Definition The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:
for right endpoint method:
$A=\lim_{n\to+\infty} R_n=\lim_{n\to+\infty}[f(x_1)\bigtriangleup x+f(x_2)\bigtriangleup x+...+f(x_n)\bigtriangleup x]$

$A=\lim_{n\to+\infty} L_n=\lim_{n\to+\infty}[f(x_0)\bigtriangleup x+f(x_2)\bigtriangleup x+...+f(x_{n-1})\bigtriangleup x]$

1) **Sample Points**
   To derive the formulas above, we choose a sample point in each interval $[x_{n-1},x_n]$, denoted as x_i^*
   Generally, we choose the sigma notation $\sum_{i=1}^{n} f(x_i)\bigtriangleup x=f(x_1)\cdot\bigtriangleup x+f(x_2)\cdot\bigtriangleup x+...+f(x_n)\cdot\bigtriangleup x$

$$A=\lim_{n\to+\infty}\sum_{i=1}^{n} f(x_i^*)\bigtriangleup x$$

2) **Midpoint Method**
   Example 3
   Let A be the area of the region that lies under the graph $f(x)=e^{-x}$ between $x=0$ and $x=1$

   a Using right endpoints, find the expression for A as a limit. Do not evaluate the liimit
   Solution:
   Since a = 0, b = 1, the width of each subinteral is

$$\bigtriangleup(x)=\frac{2-0}{n}=\frac{2}{n}$$

   Thus, the subintervals of the graph $g(x)$ are $[0,\frac{1}{n}],[\frac{1}{n},\frac{2}{n}],...,[\frac{n-1}{n},2]$; the Right Endpoint method has the sum of area approximation to be

$$\begin{aligned}
R_n &= f(x_1)\bigtriangleup x+f(x_2)\bigtriangleup x+...+f(x_n)\bigtriangleup x \\
&= e^{-x_1}\bigtriangleup x+e^{-x_2}\bigtriangleup x+...+e^{-x_n}\bigtriangleup x \\
&= e^{-\frac{2}{n}}(\frac{2}{n})+e^{-\frac{4}{n}}(\frac{2}{n})+...+e^{-\frac{2n}{n}}(\frac{2}{n})
\end{aligned}$$

Thus, the area is

$$A = \lim_{n \to +\infty} R_n = \lim_{n \to +\infty} [f(x_1) \triangle x + f(x_2) \triangle x + ... + f(x_n) \triangle x)$$

b Estimate the area by taking the sample points to the midpoints and using four subintervals and then ten subintervals

Solution:

Since n = 4, construct the subinterval for interval [0, 2], we have [0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2]. Taking the sample points, which are midpoints of the subintervals, we have
$x_1^* = 0.25, x_2^* = 0.75, x_3^* = 1.25, x_4^* = 1.75$, then we have

$$\begin{aligned} M_4 &= \sum_{i=1}^{4} f(x_i) \triangle x = f(x_1) \cdot \triangle x + f(x_2) \cdot \triangle x + ... + f(x_n) \cdot \triangle x \\ &= f(0.25) \triangle x = f(0.75) \cdot \triangle x + f(1.25) \cdot \triangle x + f(1.75) \cdot \triangle x \\ &= e^{-0.25}(0.5) + e^{-0.75}(0.5) + e^{-1.25}(0.5) + e^{-1.75}(0.5) \\ &= 0.5(e^{-0.25} + e^{-0.75} + e^{-1.25} + e^{-1.75}) \\ &= 0.8557 \end{aligned}$$

c Discussion:

If we choose n = 10, what's the area of A? Comparing to n = 4, what can you conclude?

1.4 Application

In the real world, we choose the method similar to the above methods to find distance traveled by an object during a certain period of time with the velocity given. First, read the context, and then answer questions

Time (s)	0	5	10	15	20	25	30
Velocity (mi/h)	17	21	24	29	32	31	28

After converting to the velocity to feet/sec, where 1 mi/h = 5280 / 3600 ft/s), we have a new table as following

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	46	41

By the formula **distance** = **velocity**× **time**, we would like to know

1) If time interval [0, 30], what are subintervals for the time interval?

2) When t = 0, f(0) = 25; when t = 5, $f(t)$= 31; when t =10, $f(10)$ = 35; ...

So, since taking the left endpoint of each subinterval, for the first 5 seconds, how far does the object travel? Similarly, how far does the object travel in the next 5 seconds? ... How far does the object travel in the 30 seconds? That is, what is the total distance the object has traveled?

3) If we want to increase the number of intervals, what method can we take? How many intervals will be made if we set t = 2, 0.2, 0.02.

4) Thus, we obtain a better and better estimation for the total distance as the t is arbitrary smaller and smaller, or the n is sufficient large. Please write the symbol that stands for the estimation for distance problems.

1.5 Conclusion

So far, we have learned methods for finding areas. They are, respectively, methods of right endpoint, left endpoint, and midpoint. To obtain a better estimation, we increase the number of n, and when n is larger and larger, let's assume n approaches infinite, by using limit method, we obtain a better and better estimates. Finally, we are able to answer the question about exact area under a curve after defining an integral using the limit method, which we will discuss further more in next session. Those methods such as the method of right endpoint help us find approximations for some practical problems such as distance problem.

Using limit method, can we find estimate the distance problem in a general case. For example, if we assume $v = f(t)$, where $a \leq t \leq b$ and $f(t) \geq 0$?